The Many Faces of Rationalizability

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Abstract

The rationalizability concept was introduced in Bernheim [1984] and Pearce [1984] to assess what can be inferred by rational players in a non-cooperative game in the presence of common knowledge. However, this notion can be defined in a number of ways that differ in seemingly unimportant minor details. We shed light on these differences, explain their impact, and clarify for which games these definitions coincide.

Then we apply the same analysis to explain the differences and similarities between various ways the iterated elimination of strictly dominated strategies was defined in the literature. This allows us to clarify the results of Dufwenberg and Stegeman [2002] and Chen, Long and Luo [2005] and improve upon them. We also consider the extension of these results to strict dominance by a mixed strategy.

Our approach is based on a general study of the operators on complete lattices. We allow transfinite iterations of the considered operators and clarify the need for them. The advantage of such a general approach is that a number of results, including order independence for some of the notions of rationalizability and strict dominance, come for free.

1 Introduction

1.1 Motivation

Rationalizability was introduced in Bernheim [1984] and Pearce [1984] to formalize the intuition that players in non-cooperatives games act by having common knowledge of each others' rational behaviour. Rationalizable strategies are then defined as a limit of an iterative process in which one repeatedly removes the strategies that are never best responses to the beliefs held about the other players.

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To better understand the rationale for the research here reported consider the following example.

Example 1 *Bertrand competition.*

Consider a version of Bertrand competition between two firms in which the marginal costs are 0 and in which the range of possible prices is the left-open real interval $(0, 100]$. So in this game H there are two players, each with the set $(0, 100]$ of strategies. We assume that the demand equals $100 - p$, where p is the lower price and that the profits are split in case of a tie. So the payoff functions are defined by:

$$
p_1(s_1, s_2) := \begin{cases} s_1(100 - s_1) & \text{if } s_1 < s_2 \\ \dfrac{s_1(100 - s_1)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 > s_2 \end{cases}
$$

$$
p_2(s_1, s_2) := \begin{cases} s_2(100 - s_2) & \text{if } s_2 < s_1 \\ \dfrac{s_2(100 - s_2)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_2 > s_1 \end{cases}
$$

This game has no Nash equilibrium (in pure strategies).

Consider now each player's best responses to the strategies of the opponent. Since $s_1 = 50$ maximizes the value of $s_1(100 - s_1)$ in the interval $(0, 100]$, the strategy 50 is the unique best response of the first player to any strategy $s_2 > 50$ of the second player. Further, no strategy is a best response to a strategy $s_2 \leq 50$. By symmetry the same holds for the strategies of the second player.

This eliminates for each player each strategy different than 50 and reduces the original game to the game $G := (\{50\}, \{50\}, p_1, p_2)$ in which each player has just one strategy, 50.

There are now two natural ways to proceed. If we adopt the approach of Pearce [1984], we should now focus on the *current* game G and note that $s_1 = 50$ is a best response in G to $s_2 = 50$ and symmetrically for the second player. So the iterated elimination of never best responses stops and the outcome is G.

However, if we adopt the approach of Bernheim [1984], we should continue to consider the best responses in the *original* game H. Now in H the strategy $s_1 = 49$ is a better response to $s_2 = 50$ than $s_1 = 50$ and symmetrically for the second player. So in the second round of elimination of never best responses both strategies 50 are eliminated and we reach the empty game.

So depending on the procedure we adopt we obtain two different outcomes.
□

We shall return to this example in Section 5. We shall show there that using the first elimination procedure we can still reach the empty game, if in each round *only some* strategies are removed. This might suggest that both approaches are equivalent if we do not insist on removing *all* strategies in each round (analogously to the case of iterated elimination of strictly dominated strategies). However, as we shall see, this statement is false.

So we see that the iterated elimination of best responses process can be defined in two different ways. In fact, two other definitions can be naturally envisaged. Each of these four definitions captures the original intuition in a meaningful way. Indeed, all four operators upon which these definitions rely yield the *same* outcome when applied to the original game. The differences arise when these operators are iterated.

Let us move now to the case of strict dominance. Consider the following example.

Example 2 *Production with a discontinuity.*
Consider a game H with two players, each, as in the previous example with the set $(0, 100]$ of strategies. The payoff functions are defined now by:

$$p_1(s_1, s_2) := \begin{cases} f_1(s_1, s_2) & \text{if } (s_1, s_2) \neq (100, 100) \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(s_1, s_2) := \begin{cases} f_2(s_1, s_2) & \text{if } (s_1, s_2) \neq (100, 100) \\ 0 & \text{otherwise} \end{cases}$$

where each function f_i is strictly increasing in the ith argument. (A simple example is $f_i(s_1, s_2) := s_i$.)

A possible interpretation of this game is as follows. The strategy of a player represents the amount of his resource that he chooses. If each player 'overdoes it' and chooses the maximum amount, the outcome is bad (0) for both of them. Otherwise each player gets the outcome computed by his production function f_i. Also this game has no Nash equilibrium.

Consider now each player's strategies that are not strictly dominated. Clearly every strategy $s_i \neq 100$ is strictly dominated and 100 is not strictly dominated. By eliminating all strictly dominated strategies the original game reduces to the game $G := (\{100\}, \{100\}, p_1, p_2)$ in which each player has just

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one strategy, 100. The payoff for each player to the joint strategy $(100, 100)$ is 0.

Again, there are two natural ways to proceed. If we take the usual approach, adopted in numerous publications, we should focus on the *current* game, G. Since this game is solved (i.e., each player has just one strategy) the iterated elimination of strictly dominated strategies, in short IESDS, stops and the outcome is G.

However, if we adopt the approach of Milgrom and Roberts [1990, pages 1264-1265], we should continue and consider which strategies in the game G are strictly dominated (against the opponent strategies in G) in the *original* game H. Now, for each player i each strategy 100 is strictly dominated by any other strategy s_i in the game H as each s_i yields a strictly higher payoff against the strategy 100 of the opponent. So in the second round of elimination of strictly dominated strategies both strategies 100 are eliminated and we reach the empty game.

So also here, depending on the procedure we adopt we obtain two different outcomes. (A perceptive reader may notice that the elimination of never best responses also reduces the original game to $(\{100\}, \{100\}, p_1, p_2)$ and that in the original game for each player each strategy $\neq 100$ is a better response to 100 than 100. So also here the outcome of the iterated elimination of never best responses depends on the adopted procedure.) $\qquad\square$

In fact, we show that also IESDS can be defined in four natural ways and that the resulting outcomes differ.

1.2   Approach and summary of results

To analyze in a uniform way various ways of iterated elimination of strategies here considered we consider arbitary operators on complete lattices and their transfinite iterations. This allows us to prove various results concerning the rationalizability and IESDS by simply checking the properties of underlying operators. For example, order independence for specific definitions turns out to be a direct consequence of the fact that the underlying operator is monotonic.

Before we proceed let us clarify two, rather unusual aspects of our approach. First, the use of transfinite induction and ordinals in reasoning about games is rare though not uncommon. The following illustrative examples come to our mind. In Binmore [2007, Chapter 7] (and implicitly in the original version, Binmore [1991]) a proof, attributed to G. Owen, of the Minimax Theorem is given that is based on transfinite induction. Next, in Lipman [1991] transfinite ordinals are used in a study of limited rationality, while in Lipman [1994] a

two-player game is constructed for which the ω (the first infinite ordinal) and $\omega+1$ iterations of the rationalizability operator of Bernheim [1984] differ. That is, ω iterations are insufficient to reach a fixpoint. This motivates the author to study transfinite iterations of this operator. We shall return to this matter in Section 4. In turn, Heifetz and Samet [1998] show that in general arbitrary ordinals are necessary in the epistemic analysis of strategic games based on the partition spaces. Finally, as argued in Chen, Long and Luo [2005], the notion of IESDS à la Milgrom and Roberts [1990], when used for arbitrary games, also requires transfinite iterations of the underlying operator.

The mathematical reason for the use of transfinite induction is that the underlying operators, even if they are monotonic, are in general not continuous. So to reach a fixpoint, by the theorem of Tarski [1955] (which generalizes and strengthens the initial result of Knaster [1928]), one needs to consider iterations that continue beyond ω.

Second, we consider solution concepts based on the iterated elimination of strategies that can yield the empty outcomes. This is inherent in the nature of infinite games as it can then easily happen that no strategy is a best response or that each strategy is strictly dominated. The empty outcome only indicates that in some games the players have no rational strategy to choose from. Analogous problems arise if one adopts as a solution concept the set of Nash equilibria of a non-cooperative game or the core of a cooperative game. In both cases these sets can be empty. To quote from Aumann [1985]:

> My main thesis is that a solution concept should be judged more by what it does than by what it is; more by its success in establishing relationships and providing insights into the workings of the social processes to which it is applied than by considerations of a *a priori* plausibility based on its definition alone.

Our analysis allows us to conclude that in the case of arbitrary games among four ways of defining rationalizability only two, the one due Bernheim [1984] and its contracting version (a notion explained in Section 3), are meaningful. The crucial feature of these two operators is that they refer to the best responses in the initial game and *not* the currently considered game. As a result these two operators are monotonic. However, under the assumption that to each belief in the initial game a best response exists (assumption **B**), the iterations of all four operators coincide even though the other two still are not monotonic. We also explain the need for transfinite iterations of the corresponding operators, even for the games that satisfy assumption **B**. Assumption **B** is in particular satisfied by the compact games with contin-

uous payoff functions, which explains why the reported differences were not discussed in the literature.

We also apply the same analysis to the notion of strict dominance. We explain that the operator underlying the usual definiton is not monotonic. This clarifies why this definition of strict dominance is not order independent for arbitrary infinite games (see Dufwenberg and Stegeman [2002]).

In contrast, the version of the iterated elimination of strictly dominated strategies used in Milgrom and Roberts [1990] and more recently in Ritzberger [2001, Section 5.1] is monotonic. The contracting version of it studied in Chen, Long and Luo [2005]. The authors of the latter paper show that the resulting elimination procedure is stronger than the customary one, requires transfinite iterations, does not remove any Nash equilibria, and is order independent. In our framework order independence of both versions is a direct consequence of the monotonicity of the underlying operators.

We also consider a natural condition $\mathbf{C}(\alpha)$ of the initial game, parametrized by an ordinal α, that formalizes the statement that in each reduction reachable from the initial game in α iterations every strictly dominated strategy has an undominated dominator. We show that $\forall \alpha \mathbf{C}(\alpha)$ implies order independence of the usual definition of strict dominance. Further, $\forall \alpha \mathbf{C}(\alpha)$ implies that the iterations of all four considered operators coincide. Assumption $\forall \alpha \mathbf{C}(\alpha)$ is in particular satisfied by the finite games.

Our formalization of the condition $\mathbf{C}(\alpha)$ differs from the one used in Dufwenberg and Stegeman [2002] for which the authors showed that the iterated elimination of strictly dominated strategies may fail to be order independent in their sense. (Our definitions of order independence differ, since we allow transfinite iterations of the underlying operators.) We show that $\mathbf{C}(\omega)$ implies order independence, in the sense used in Dufwenberg and Stegeman [2002], of strict dominance. This yields a minor improvement of their result. Finally, we explain how to extend these results to the case of iterated elimination of strictly dominated strategies by a mixed strategy.

1.3 Plan of the paper

The paper is organized as follows. Standard concepts on strategic games and examples of belief structures to which the results of this paper apply are introduced in Section 2. Next, in Section 3 some general results about monotonic and contracting operators on complete lattices are established that provide a basis for our approach.

In Section 4 we discuss two operators that underly the definitions of rationalizability, including the one due to Bernheim [1984]. Both are monotonic,

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so the revelant properties of these operators and of their outcomes are direct
consequences of the general results established in Section 3.

Then in Section 5 we discuss two slightly different operators defining a no-
tion of rationalizability. We conclude that one of them cannot be meaningfully
used to formalize the notion of rationalizablity and the other one, underly-
ing the definition of rationalizability due to Pearce [1984], leads to too weak
conclusions for a game modelling Bertrand competition for two firms. On the
other hand, in Section 6 we show that the best response property in the initial
game implies that the iterations of all four operators coincide.

Then in Sections 7 and 8 we discuss in detail four natural ways of defining
iterated elimination of strictly dominated strategies and analyze when the it-
erations of the corresponding four operators coincide. In Section 9 we compare
our results with those of Dufwenberg and Stegeman [2002] and clarify in what
sense we established a new order independence result.

Next, in Section 10 consider an extension of these results to the case of
strict dominance by a mixed strategy. Finally, in Section 11 we assess the
results of this paper stressing that monotonicity and transfinite iterations are
most relevant for a study of reduction operators on games.

In the literature we found only one reference concerned with a similar
comparative analysis of the notion of rationalizability. Ambroszkiewicz [1994]
studied the limited case of two-player games and beliefs being equal to the
strategies of the opponent, and showed that the finite iterations of two opera-
tors concerned with the notion of rationalizablity coincide for compact games
with continuous payoffs. We clarify his result in Section 6.

Parts of this research were reported in Apt [2005].

2 Game theoretic preliminaries

2.1 Strategic games and their restrictions

Let us move now to the subject of strategic games. Given n players $(n > 1)$
we represent a strategic game (in short, a game) by a sequence

$$(S_1, \ldots, S_n, p_1, \ldots, p_n),$$

where for each $i \in [1..n]$

- S_i is the non-empty set of **strategies** available to player i,

- p_i is the **payoff function** for the player i, so $p_i : S_1 \times \ldots \times S_n \to \mathcal{R}$,
  where $\mathcal{R}$ is the set of real numbers.

Given a sequence of sets of strategies $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i and use the following standard notation:

- $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$,

- $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$.

We denote the strategies of player i by s_i, possibly with some superscripts.

We say that $G := (S_1, \ldots, S_n)$ is a **restriction** of a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ if each S_i is a (possibly empty) subset of T_i. We identify the restriction $(T_1, \ldots, T_n)$ with H. The restrictions are naturally ordered by the componentwise set inclusion:

$$(S_1, \ldots, S_n) \subseteq (S'_1, \ldots, S'_n) \text{ iff } S_i \subseteq S'_i \text{ for all } i \in [1..n].$$

If some S_i is empty, $(S_1, \ldots, S_n, p_1, \ldots, p_n$ (we identify here each p_i with its restriction to the smaller domain) is not a game and the references to $p_j(s)$ (for any $j \in [1..n]$) are incorrect, so we shall need to be careful about this. If all S_i are empty, we call G an **empty restriction**.

2.2 Belief structures

Throughout the paper $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ is a *fixed* game.

Our intention is to explain various concepts of rationalizability abstracting from specific sets of beliefs that are assumed. Therefore we only assume that each player i in the game H has some further unspecified non-empty set of beliefs $\mathcal{B}_i$ about his opponents and that each payoff function p_i can be modified to an **expected payoff** function $p_i : S_i \times \mathcal{B}_i \to \mathcal{R}$.

In what follows we also assume that each set of beliefs $\mathcal{B}_i$ of player i in H can be *narrowed* to any restriction G of H. We denote the outcome of this **narrowing** of $\mathcal{B}_i$ to G by $\mathcal{B}_i \mathbin{\dot{\cap}} G$. The set $\mathcal{B}_i \mathbin{\dot{\cap}} G$ can be viewed as the set of beliefs of player i in the restriction G. We call then the pair $(\mathcal{B}, \dot{\cap})$, where $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$, a **belief structure** in the game H.

The following natural property of a belief structure $(\mathcal{B}, \dot{\cap})$ in H will be relevant.

A If $G_1 \subseteq G_2 \subseteq H$, then for all $i \in [1..n]$, $\mathcal{B}_i \mathbin{\dot{\cap}} G_1 \subseteq \mathcal{B}_i \mathbin{\dot{\cap}} G_2$.

This property simply states that for each player the set of his beliefs in a restriction G_1 of G_2 is a subset of the set of his beliefs in G_2.

The following four belief structures were considered in the literature. Given a finite non-empty set A we denote here by ΔA the set of probability distributions over A and by $\Delta^{\circ} A$ the set of probability distributions over A that assign a positive probability to each element of A.

(i) $\mathcal{B}_i := T_{-i}$ for $i \in [1..n]$.

So beliefs are joint pure strategies of the opponents (usually called **point beliefs**). For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := S_{-i}.$$

We call then $(\mathcal{B}, \dot{\cap})$ the **pure** belief structure in H. This belief structure was considered in Bernheim [1984].

A specific case with a different definition of $\dot{\cap}$ was considered in Pearce [1984]. In that paper H is a mixed extension of a finite game. So given initial finite sets of strategies $I_1, \ldots, I_n$ each set T_i equals ΔI_i, i.e., $H := (\Delta I_1, \ldots, \Delta I_n, p_1, \ldots, p_n)$. Then for a restriction $G := (S_1, \ldots, S_n)$ of H

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := \Pi_{j \neq i} \overline{S_j},$$

where for a set S_j of mixed strategies of player j, $\overline{S_j}$ denotes its convex hull.

(ii) Assume H is finite. $\mathcal{B}_i := \Pi_{j \neq i} \Delta T_j$ for $i \in [1..n]$.

So beliefs are joint mixed strategies of the opponents. For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := \Pi_{j \neq i} \Delta S_j.$$

This belief structure was considered in Bernheim [1984].

(iii) Assume H is finite. $\mathcal{B}_i := \Delta T_{-i}$ for $i \in [1..n]$.

So beliefs are probability distributions over the set of joint pure strategies of the opponents (usually called **correlated mixed strategies**). For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \mathbin{\dot{\cap}} G := \Delta S_{-i}.$$

This belief structure was mentioned in Bernheim [1984] and studied in Brandenburger and Dekel [1987], where the term **correlated rationalizability** was introduced.

(iv) Assume H is finite. $\mathcal{B}_i := \Pi_{j \neq i} \Delta^\circ T_j$ for $i \in [1..n]$.

So beliefs are joint **totally mixed** strategies of the opponents. For a restriction $G := (S_1, \ldots, S_n)$ of H we define

$$\mathcal{B}_i \dot{\cap} G := \Pi_{j \neq i} \Delta^\circ S_j.$$

This belief structure was studied in Pearce [1984], where a best response to a belief formed by a joint totally mixed strategy of the opponents is called a **cautious response**.

Given two finite sets A and B such that $A \subseteq B$, we identify each probability distribution on A with the probability distribution on B in which 0 is assigned to each element in $B \setminus A$. Then $A \subseteq B$ implies $\Delta A \subseteq \Delta B$. It is now straightforward to see that property **A** is satisfied by the belief structures from examples (i)—(iii). So by appropriately choosing the belief structure $(\mathcal{B}, \dot{\cap})$ we shall be able to apply our results to a variety of frameworks including the ones considered in Bernheim [1984] and Pearce [1984].

It is important, however, to note that property **A** is not satisfied by the belief structure from example (iv). The reason is that for finite sets A and B such that $A \subseteq B$ the inclusion $\Delta^\circ A \subseteq \Delta^\circ B$ does not hold. So the results of our paper do not apply to the modifications of the notion of rationalizability that rely on totally mixed strategies, for example Herings and Vannetelbosch [2000], where the notion of weak perfect rationalizability is studied.[1]

3   Operators on complete lattices

We are interested in iterated reductions of strategic games entailed by various ways of removing strategies. To deal with them in a uniform way we define the relevant concepts for arbitrary operators on a fixed complete lattice $(D, \subseteq)$ with the largest element $\top$.

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[1]This is not surprising since by the result of Pearce [1984] the notions of rationalizability w.r.t. totally mixed strategies of the opponent and of not being weakly dominated by a mixed strategy coincide in two-player finite games and, as is well-known, the latter notion fails to be order independent.

In what follows we use ordinals and denote them by α, β, γ. Given a, possibly transfinite, sequence $(G_\alpha)_{\alpha < \gamma}$ of elements of D we denote their join and meet respectively by $\bigcup_{\alpha < \gamma} G_\alpha$ and $\bigcap_{\alpha < \gamma} G_\alpha$.

In the subsequent applications D will be the set of all restrictions of a fixed strategic game H for n players, ordered by the componentwise set inclusion $\subseteq$ (so H is the largest element) and $\bigcup_{\alpha < \gamma}$ and $\bigcap_{\alpha < \gamma}$ the customary set-theoretic operations on them. But this additional information on the structure of D will be irrelevant in the remainder of this section.

We now establish some general results on operators on complete lattices. The proofs are straightforward and the results readily apply to the operators we shall subsequently study.

Definition 1 Let T be an operator on $(D, \subseteq)$, i.e., $T : D \to D$.

- We say that an element G is a **fixpoint** of T if $T(G) = G$.

- We call T **monotonic** if for all G_1, G_2

$$G_1 \subseteq G_2 \text{ implies } T(G_1) \subseteq T(G_2).$$

- We call T **contracting** if for all G

$$T(G) \subseteq G.$$

- We define the **contracting version** of T by:

$$\overline{T}(G) := T(G) \cap G.$$

- We define by transfinite induction a sequence of elements T^α of D, where α is an ordinal, as follows:

  - $T^0 := \top$,
  - $T^{\alpha+1} := T(T^\alpha)$,
  - for all limit ordinals β, $T^\beta := \bigcap_{\alpha < \beta} T^\alpha$.

- We call the least α such that $T^{\alpha+1} = T^\alpha$ the **closure ordinal** of T and denote it by α_T. We call then T^{α_T} the **outcome of** (iterating) T. $\quad\square$

The outcome of an operator can be the least element of the complete lattice. So in the case of the complete lattice of the restrictions of H ordered by the componentwise set inclusion $\subseteq$ the outcome can be an empty restriction. In general the closure ordinal of T, and hence its outcome, do not need to exist. (Take for example an operator oscillating between two values.) However, we have the following classic result due to Tarski [1955].[2]

Tarski's Fixpoint Theorem Every monotonic operator on $(D, \subseteq)$ has a largest fixpoint. This fixpoint is the outcome of T, i.e., it is of the form T^{α_T}.

Clearly, if T is monotonic, then so is $\overline{T}$. Other observations concerning the above notions are gathered in the following note.

Note 1 *Consider an operator T on $(D, \subseteq)$.*

  (i) If T is contracting or monotonic, then $T^{\alpha+1} \subseteq T^\alpha$ for all α.

  (ii) Suppose that $T^{\alpha+1} \subseteq T^\alpha$ for all α. Then

  - $\overline{T}^\alpha = T^\alpha$ *for all α,*
  - *the outcomes of T and $\overline{T}$ exist and coincide.*

Proof. The existence of the outcome of T under the assumption of (ii) follows by the standard arguments of set theory (see, e.g., Aczel [1977]). The other claims follow by transfinite induction. □

So if we are only interested in the operator iterations or its outcome, it does not matter whether we choose an operator T that is contracting or monotonic, or its contracting version $\overline{T}$.

In what follows we shall frequently use the following lemma.

Lemma 1 *Consider two operators T and R on $(D, \subseteq)$ such that*

  - *for all α, $T(R^\alpha) \subseteq R(R^\alpha)$,*
  - *T is monotonic.*

Then for all α

$$T^\alpha \subseteq R^\alpha.$$

In particular if R has an outcome, then $T^{\alpha_T} \subseteq R^{\alpha_R}$.

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[2]We use here its 'dual' version in which the iterations start at the largest and not at the least element of a complete lattice.

Proof. We prove the first claim by transfinite induction. By the definition of the iterations we only need to consider the induction step for a successor ordinal. So suppose the claim holds for some α. Then by the first two assumptions and the induction hypothesis we have the following string of inclusions and equalities:

$$T^{\alpha+1} = T(T^\alpha) \subseteq T(R^\alpha) \subseteq R(R^\alpha) = R^{\alpha+1}.$$

By Tarski's Fixpoint Theorem T has an outcome, so the second claim follows immediately from the first. $\qquad\square$

We now formalize the idea that a procedure of iterated elimination of strategies is order independent. Interestingly, it is possible to state this property without specializing the underlying complete lattice to that of all restrictions of the initial game, by simply focusing on iterations of operators. The definition we provide also takes care of the possibility that the elimination process takes more than ω iteration steps.

Definition 2 Consider a contracting operator T ('T removes strategies') on $(D, \subseteq)$. We say that T is **order independent** if

$$R^{\alpha_R} = T^{\alpha_T}$$

('the outcomes of the iterated eliminations of strategies coincide') for each operator R such that for all α

- $T(R^\alpha) \subseteq R(R^\alpha) \subseteq R^\alpha$

  ('R removes from R^α only strategies that T removes')

- if $T(R^\alpha) \subsetneq R^\alpha$, then $R(R^\alpha) \subsetneq R^\alpha$

  ('if T can remove some strategies from R^α, then R as well').

(Note that R^{α_R} exists by Note 1.) $\qquad\square$

The intuitions provided in the brackets hopefully make this definition self-explanatory. This definition is perfectly satisfactory as long as we consider procedures that *remove* strategies. However, we shall consider here arbitrary procedures on games and sometimes they may *add* strategies. To deal with this more general setting we introduce a definition of order independence for arbitrary operators.

Definition 3 Consider an operator T on $(D, \subseteq)$.

- We say that the operator R is a **relaxation** of T if for all α

  - $T(R^\alpha) \subseteq R(R^\alpha)$,
  - if $T(R^\alpha) \subseteq R^\alpha$, then $R(R^\alpha) \subseteq R^\alpha$,
  - if R^α is a fixpoint of R, then it is a fixpoint of T.

- We say that T is **order independent** if the set

$$\{G \mid G \text{ is an outcome of a relaxation of } T\}$$

  has at most one element. □

The intuition behind the conditions defining a relaxation is as follows. Suppose that T is a procedure on the set of restrictions of the initial game. Then the first condition states that during the iterations of R (i.e., for all restrictions R^α) the operator R 'approximates' T from above. In turn, the second condition states that if T removes some strategies from some restriction R^α, then so does R. Finally, the third condition states that if some strategy can be removed from/added to some restriction R^α by the operator T (R^α is not a fixpoint of T), then the same holds for the R operator (R^α is not a fixpoint of R).

Note that this definition of order independence of an operator T does not even require that T has an outcome. If it does have, then each relaxation of it has the same outcome as T, if it has one.

The following observation shows that the second definition generalizes the first one.

Note 2 *For a contracting operator T on $(D, \subseteq)$ both definitions of order independence coincide.*

Proof. For a contracting operator T the first two conditions of the definition of a relaxation R are equivalent to the first condition on R in Definition 2, while the third condition is equivalent to the second condition on R in Definition 2.

Moreover, for all relaxations R of T we have $R^{\alpha+1} \subseteq R^\alpha$ for all α and hence, by Note 1, they all have an outcome. Also T is a relaxation of itself, so order independence in the sense of Definition 3 is equivalent to the statement that $R^{\alpha_R} = T^{\alpha_T}$ holds for all relaxations R of T. □

From now on when referring to order independence we shall mean the latter definition. The following general result holds.

Theorem 1 (Order Independence) *Every monotonic operator on $(D, \subseteq)$ is order independent.*

Proof. Let T be a monotonic operator. By Tarski's Fixpoint Theorem T has an outcome, T^{α_T}, which is the largest fixpoint of T. To prove the theorem it suffices to show that

$$\{G \mid G \text{ is an outcome of a relaxation of } T\} = \{T^{\alpha_T}\}.$$

So take a relaxation R of T that has an outcome, R^{α_R}. By Lemma 1 $T^{\alpha_T} \subseteq R^{\alpha_R}$. But by the definition of a relaxation R^{α_R} is a fixpoint of T and T^{α_T} is the largest fixpoint of T, so also $R^{\alpha_R} \subseteq T^{\alpha_T}$. Consequently $R^{\alpha_R} = T^{\alpha_T}$.
$\square$

Intuitively, this result can be interpreted as follows. Suppose that an operator T removes *all* strategies from a restriction of a game H that meet some property and that we iterate this operator starting with H. Suppose now that at each stage we remove only *some* strategies that meet this property (instead of all). Then, when T is monotonic, we still end up with the same outcome (which is the largest fixpoint of the T operator).

Note that we did not assume here that the operator T is contracting and consequently had to rely on the second definition of order independence. Also, in the proof we did not use the second condition of the definition of a relaxation. In our approach this condition is needed only to deal with the contracting operators that are not monotonic.

4 The GR and $\overline{GR}$ operators

We now apply the above considerations to specific operators. Each of them is defined in the context of a fixed game H and a belief structure $(\mathcal{B}, \dot{\cap})$ in H.

Given a restriction $G := (S_1, \ldots, S_n)$ of H and a belief $\mu_i \in \mathcal{B}_i \dot{\cap} G$ we say that a strategy s_i of player i in the game H is a ***best response to*** μ_i ***in*** G, and write $s_i \in BR_G(\mu_i)$, if

$$\forall s_i' \in S_i \; p_i(s_i, \mu_i) \geq p_i(s_i', \mu_i).$$

Note that s_i does not need to be an element of S_i.

We now introduce the following operator GR (standing for 'global rationalizability') on the set of restrictions of H:

$$GR(G) := (S_1', \ldots, S_n'),$$

where for all $i \in [1..n]$

$$S_i' := \{s_i \in T_i \mid \exists \mu_i \in \mathcal{B}_i \cap G \ s_i \in BR_H(\mu_i)\}.$$

So $GR(G)$ is obtained by removing from H all strategies that are never best responses *in* H (note this reference to H) to a belief held in G. That is, when removing the strategies we allow justifications (for their removal) from the initial game H.

Thanks to property **A** the operator GR is monotonic. So we can apply to GR and its contracting version $\overline{GR}$ Note 1 and as a result we can confine further analysis to the latter operator. The GR operator was considered in Bernheim [1984] (called there λ, see page 1015).[3] On the account of the results from the previous section the $\overline{GR}$ operator enjoys various properties. We list them as the following result.

Theorem 2 ($\overline{GR}$)

 (i) *The largest fixpoint of $\overline{GR}$ exists and is its outcome.*

 (ii) *$\overline{GR}$ is order independent.*

 (iii) *For all relaxations R of $\overline{GR}$ and all α we have $\overline{GR}^\alpha \subseteq R^\alpha$.*   □

In general, infinite iterations of $\overline{GR}$ can be necessary. In fact, in some games ω iterations of $\overline{GR}$ are insufficient to reach the outcome, that is, the closure ordinal of $\overline{GR}$ can be larger than ω.

Example 3 Consider the following game H with two players. The set of strategies for each player is the set of natural numbers $\mathcal{N}$ augmented by -1. The payoff functions are defined as follows. For $k, \ell \geq 0$ we put:

$$p_1(k, \ell) := \begin{cases} \ell + 1 & \text{if } k = \ell + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(k, \ell) := \begin{cases} k & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

For the remaining pairs of strategies we put for $k, \ell \geq -1$ and $k_0, \ell_0 \geq 0$:

$$p_1(-1, \ell) := \ell + 1,$$
$$p_1(k_0, -1) := k_0,$$
$$p_2(k, -1) := k,$$
$$p_2(-1, \ell_0) := \ell_0.$$

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[3]The reader may now notice that in Example 1 we used the $\overline{GR}$ operator instead of GR, which is more complicated to justify informally.

Further we assume the pure belief structure, i.e., the beliefs are the strategies of the opponent. The following two diagrams explain the structure of this game. In each of them on the left we depict strategies of player 1 and on the right the strategies of player 2. An arrow from a to b stands for the statement 'strategy a is a best response to strategy b'. In particular, no arrow leaves strategy 0 of player 1, which indicates that it is not a best response to any strategy of player 2.
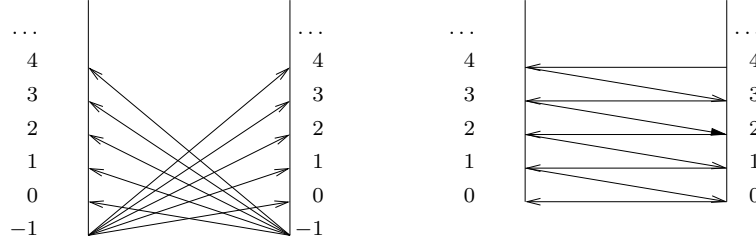
Note that

- each -1 is a best response to any non-negative strategy,

- no best response to any -1 strategy exists,

- the best response to strategy $k \geq 0$ of player 1 is k,

- the best response to strategy $\ell \geq 0$ of player 2 is $\ell + 1$.

Denote $\mathcal{N} \cup \{-1\}$ by $\mathcal{N}'$. It is easy to see that

$$\overline{GR}^0 = (\mathcal{N}', \mathcal{N}'),$$
$$\overline{GR}^1 = (\mathcal{N}' \setminus \{0\}, \mathcal{N}'),$$
$$\overline{GR}^2 = (\mathcal{N}' \setminus \{0\}, \mathcal{N}' \setminus \{0\}),$$
$$\overline{GR}^3 = (\mathcal{N}' \setminus \{0, 1\}, \mathcal{N}' \setminus \{0\}),$$
$$\overline{GR}^4 = (\mathcal{N}' \setminus \{0, 1\}, \mathcal{N}' \setminus \{0, 1\}),$$
$$\cdots$$

So $\overline{GR}^\omega = (\{-1\}, \{-1\})$. But in the initial game no -1 is a best response to 'the other' -1. So $\overline{GR}((\{-1\}, \{-1\})) = (\emptyset, \emptyset)$, that is $\overline{GR}^{\omega+1} = (\emptyset, \emptyset)$. $\quad \square$

As already mentioned in Section 1, in Lipman [1994] a two-player game is constructed for which ω iterations of GR are insufficient to reach the outcome. In this example each set of beliefs $\mathcal{B}_i$ consists of the mixed strategies of the opponent and the game is considerably more complex.

5 The *LR* and $\overline{LR}$ operators

In this section we analyze the following operator LR (standing for 'local rationalizability') on the set of restrictions of H:

$$LR(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \exists \mu_i \in \mathcal{B}_i \cap G \; s_i \in BR_G(\mu_i)\}.$$

So $LR(G)$ is obtained by removing from H all strategies that are never best responses *in* G (so *not* in H) to a belief held in G. That is, when removing strategies we use justifications (for their removal) from the currently considered game G. For each restriction G of H, $s_i \in BR_H(\mu_i)$ implies $s_i \in BR_G(\mu_i)$, so for all restrictions G we have $GR(G) \subseteq LR(G)$.

Unfortunately, the LR operator cannot be used as a meaningful basis for the rationalizability notion.

Example 4 To see this reconsider the game from Example 3. We first prove by induction that for $n \geq 0$ we have $LR^n = \overline{GR}^n$.

The base case obviously holds. Denote the set $\{0, \ldots, m\}$ by I_m. For the induction step note that if $n = 2k$, then

$$LR^{2k} = \overline{GR}^{2k} = (\mathcal{N}' \setminus I_k, \mathcal{N}' \setminus I_k).$$

For any strategy $m > k$ we have both

$$p_1(m+1, m) = m + 1 > 0 = p_1(k+1, m)$$

and

$$p_1(m+1, -1) = m + 1 > k + 1 = p_1(k+1, -1).$$

This shows that for each strategy $\ell \in \mathcal{N}' \setminus I_k$ of player 2 the strategy $k+1$ of player 1 is not a best response to ℓ in LR^{2k} and consequently

$$LR^{2k+1} = (\mathcal{N}' \setminus I_{k+1}, \mathcal{N}' \setminus I_k) = \overline{GR}^{2k+1}.$$

A similar argument applies when $n = 2k + 1$.

Consequently $LR^\omega = GR^\omega = (\{-1\}, \{-1\})$. But for all $s_1, s_2 \in \mathcal{N}'$ we have $p_1(s_1, -1) \geq p_1(-1, -1) = 0$ and $p_2(-1, s_2) \geq p_2(-1, -1) = -1$. So for each player each strategy from $\mathcal{N}'$ is a best response to the (unique) strategy -1 of the other player in LR^ω. Hence $LR^{\omega+1} = (\mathcal{N}', \mathcal{N}')$. This shows that LR has no outcome. Also it is not monotonic and not even contracting. $\square$

Consequently, we confine our attention to its contracting version, the $\overline{LR}$ operator (defined by $\overline{LR}(G) := LR(G) \cap G$), that always has an outcome. Note for example that for the game from Example 3 we have $\overline{LR}^{\omega} = (\{1\}, \{1\}) = \overline{LR}^{\omega+1}$, so $(\{-1\}, \{-1\})$ is the outcome of $\overline{LR}$. This operator was introduced implicitly and for specific games, in Pearce [1984], in Definition 1 on page 1032.

Unfortunately, $\overline{LR}$ fails to satisfy any of the properties of $\overline{GR}$ listed in the $\overline{GR}$ Theorem 2. First, the largest fixpoint of $\overline{LR}$ does not need to exist. Indeed, let us return to the game from Example 3. For all $x \in \mathcal{N} \cup \{-1\}$ the restriction $(\{x\}, \{x\})$ is a fixpoint of $\overline{LR}$. However, their union, H, is not. Consequently, by Tarski's Fixpoint Theorem $\overline{LR}$ is not monotonic.

Next, we exhibit a relaxation R of $\overline{LR}$ the outcome of which is different than the outcome of $\overline{LR}$.

Example 5 Consider a two-player game H in which the set of strategies for each player is the set $\mathcal{N}$ of natural numbers. The payoff to each player is the number (strategy) he selected. Take as the set of mixed strategies the probability distributions on $\mathcal{N}$ with a finite support (sometimes called simple probabilities). So when computing the payoffs to mixed strategies each of them can be written as $\sum_{i \in A} \pi(i) \cdot i$, where A is a finite subset of $\mathcal{N}$ and π is a probability distribution on A. (We shall return to this definition in Section 10.) Take the belief structure in which the beliefs of each player is the set of so defined mixed strategies of the opponent.

Clearly, no strategy is a best response to a mixed strategy of the opponent. So $\overline{LR}(H) = (\emptyset, \emptyset)$ and $(\emptyset, \emptyset)$ is the outcome of $\overline{LR}$. However, for any $i \geq 0$ the restriction $(\{i\}, \{i\})$ is the outcome of a relaxation R of $\overline{LR}$ such that $R(H) = (\{i\}, \{i\})$. $\qquad\square$

This example shows that the rationalizability notion entailed by the $\overline{LR}$ operator is not order independent. In this example the outcome of a relaxation of $\overline{LR}$ is strictly larger than the outcome of $\overline{LR}$. A more troublesome example is the following one in which the outcome of a relaxation of $\overline{LR}$ is strictly smaller than the outcome of $\overline{LR}$.

Example 6 We return here to Example 1 from Section 1. So H is a game with two players, each with the set $(0, 100]$ of strategies, and the payoff functions are defined by:

$$p_1(s_1, s_2) := \begin{cases} s_1(100 - s_1) & \text{if } s_1 < s_2 \\ \dfrac{s_1(100 - s_1)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 > s_2 \end{cases}$$

$$p_2(s_1, s_2) := \begin{cases} s_2(100 - s_2) & \text{if } s_2 < s_1 \\ \dfrac{s_2(100 - s_2)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_2 > s_1 \end{cases}$$

Also, we assume the pure belief structure.

We noted already in Example 1 that $\overline{LR}(H) = (\{50\}, \{50\})$ and $\overline{LR}((\{50\}, \{50\})) = ((\{50\}, \{50\}))$.

Take now the relaxation R of $\overline{LR}$ defined by:

$$R(G) := \begin{cases} ((0, 50], (0, 50]) & \text{if } G = H \\ \overline{LR}(G) & \text{otherwise} \end{cases}$$

Then $R(H) = (0, 50], (0, 50])$ and $R(((0, 50], (0, 50])) = (\emptyset, \emptyset)$ since $\overline{LR}(((0, 50], (0, 50])) = (\emptyset, \emptyset)$. This shows that H cannot be reduced to an empty restriction by $\overline{LR}$ though it can by some relaxation of it.

Finally, note that in this example also the LR operator exhibits an erratic behaviour. Indeed, $LR(H) = (\{50\}, \{50\})$, i.e., $LR^1 = (\{50\}, \{50\})$. However, it is easy to see that $(49, 49) \in LR(\{50\}, \{50\})$, since $p_1(49, 50) > p_1(50, 50)$ and $p_2(50, 49) > p_2(50, 50)$. So LR^2 is not a restriction of LR^1. $\qquad\square$

This example suggest that $\overline{LR}$ is unnecessarily weak. Also, it is counterintuitive that when $\overline{LR}$ is used to define rationalizability, in some natural games it is beneficial to eliminate at certain stages only *some* strategies that are never best responses.

It is also easy to see that in Example 6 the outcome of the $\overline{GR}$ operator is an empty restriction. Indeed, $\overline{GR}(H) = (\{50\}, \{50\})$ and $\overline{GR}((\{50\}, \{50\})) = (\emptyset, \emptyset)$, since $s_1 = 49$ is a better response in H to $s_2 = 50$ than $s_1 = 50$ and symmetrically for the second player. So $\overline{GR}$ is strictly stronger than $\overline{LR}$ in the sense that its outcome can be strictly smaller than that of $\overline{LR}$.

In this example the outcome of $\overline{GR}$ is the same as the outcome of a relaxation of $\overline{LR}$. However, this does not need to hold in general. In other words, $\overline{GR}$ defines a genuinely more powerful rationalizability notion that cannot be derived from $\overline{LR}$.

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Indeed, reconsider the game from Example 3. It is easy to see that the finite iterations of $\overline{LR}$ and $\overline{GR}$ coincide, so $\overline{LR}^\omega = (\{-1\}, \{-1\})$. But $(\{-1\}, \{-1\})$ is a fixpoint of $\overline{LR}$, so using $\overline{LR}$ we cannot reduce the initial game to an empty restriction. Moreover, the same holds for any relaxation R of $\overline{LR}$. Indeed, it is easy to see that for each relaxation R of $\overline{LR}$ its outcome always contains the joint strategy $(-1, -1)$.

6   When the outcomes of $\overline{GR}$ and $\overline{LR}$ coincide

To analyze the situations when the iterations of the $\overline{GR}$ and $\overline{LR}$ operators coincide we introduce the following property of the initial game H, given a belief structure $(\mathcal{B}, \dot\cap)$ in H.

B For all beliefs $\mu_i \in \mathcal{B}_i$ a best response to μ_i in H exists.

The importance of this property stems from the fact that it allows us to equate during each iteration of the $\overline{LR}$ operator the best responses in the 'current' game with the best responses in the initial game.

For the finite games and all belief structures property **B** obviously holds but it can clearly fail for the infinite games. For instance, it does not hold in the game considered in Example 5 since in this game no strategy is a best response to a strategy of the opponent. Also it does not hold in the game from Example 3 but we now offer an example of a game in which property **B** does hold and in which the iterations of length $> \omega$ of both $\overline{GR}$ and $\overline{LR}$ are still unavoidable.

Example 7 We modify the game from Example 3 by adding the third player and by removing the strategy -1 from the set of strategies of each player. So the set of strategies for each player is the set of natural numbers $\mathcal{N}$. The payoff functions are now defined as follows:

$$p_1(k, \ell, m) := \begin{cases} \ell + 1 & \text{if } k = \ell + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(k, \ell, m) := \begin{cases} k & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

$$p_3(k, \ell, m) := 0.$$

Further we assume the pure belief structure. Note that

- the best response to $s_{-1} = (\ell, m)$ is $\ell + 1$,

- the best response to $s_{-2} = (k, m)$ is k,

- each $m \in \mathcal{N}$ is a best response to $s_{-3} = (k, \ell)$.

So to each joint strategy of the opponents a best response exists. That is, property **B** is satisfied. Analogously to Example 3 we have:

$$\begin{aligned}
\overline{GR}^0 &= (\mathcal{N}, \mathcal{N}, \mathcal{N}), \\
\overline{GR}^1 &= (\mathcal{N} \setminus \{0\}, \mathcal{N}, \mathcal{N}), \\
\overline{GR}^2 &= (\mathcal{N} \setminus \{0\}, \mathcal{N} \setminus \{0\}, \mathcal{N}), \\
\overline{GR}^3 &= (\mathcal{N} \setminus \{0, 1\}, \mathcal{N} \setminus \{0\}, \mathcal{N}), \\
\overline{GR}^4 &= (\mathcal{N} \setminus \{0, 1\}, \mathcal{N} \setminus \{0, 1\}, \mathcal{N}),
\end{aligned}$$

. . .

So $\overline{GR}^\omega = (\emptyset, \emptyset, \mathcal{N})$. Also $\overline{GR}((\emptyset, \emptyset, \mathcal{N})) = (\emptyset, \emptyset, \emptyset)$, so $\overline{GR}^{\omega+1} = (\emptyset, \emptyset, \emptyset)$.

Finally, it is easy to see that $\overline{LR}^\alpha = \overline{GR}^\alpha$ for all α (this is in fact, a consequence of a result proved below), so $\omega + 1$ is the closure ordinal of $\overline{LR}$, as well. □

For infinite games a natural situation when property **B** holds is the following. Recall that a game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ is called ***compact*** if the strategy sets are compact subsets of a complete metric space, and ***own-uppersemicontinuous*** if each payoff function p_i is uppersemicontinuous in the ith argument. In turn, p_i is called ***uppersemicontinuous in the ith argument*** if the set $\{s_i' \in T_i \mid p_i(s_i', s_{-i}) \geq r\}$ is closed for all $r \in \mathcal{R}$ and all $s_{-i} \in T_{-i}$.

As explained in Dufwenberg and Stegeman [2002] (see the proof of Lemma on page 2012) for such games and pure belief structures property **B** holds by virtue of a standard result from topology. If we impose a stronger condition on the payoff functions, namely that each of them is continuous —the game is called then ***continuous***— then we are within the framework considered in Bernheim [1984]. As shown there for compact and continuous games iterations of length $> \omega$ for the GR operator do not need to be considered, that is, its outcome (or equivalently the outcome of $\overline{GR}$) can be reached in ω iterations. To put it more succinctly: $\alpha_{GR} \leq \omega$.

In the presence of properties **A** and **B** still many differences between the operators $\overline{GR}$ and $\overline{LR}$ remain. In particular, $\overline{LR}$ does not need to be monotonic and its largest fixpoint does not need to exist. Indeed, consider the following example in which we consider a finite game, which ensures property **B**.

Example 8 Consider the game $H := (A, A, p_1, p_2)$, where $A = \{1, \ldots, n\}$ and for $i, j \in A$

$$p_1(i, j) := i,$$
$$p_2(i, j) := 1.$$

Assume the belief structure in which the beliefs of each player is the set of the mixed strategies of the opponent, i.e., $\mathcal{B}_1 = \mathcal{B}_2 = \Delta A$.

Note that

- n is the unique best response of player 1 to any belief $\mu_1 \in \mathcal{B}_1$ about player 2,

- each $j \in A$ is a best response of player 2 to any belief $\mu_2 \in \mathcal{B}_2$ about player 1.

So $\overline{LR}(H) = (\{n\}, A)$ and $(\{n\}, A)$ is a fixpoint of $\overline{LR}$. Also each restriction $(\{j\}, \{j\})$ for $j \in A$ is a fixpoint of $\overline{LR}$. However, their union, H, is not.

Furthermore $(\{1\}, \{1\}) \subseteq H$, but $\overline{LR}(\{1\}, \{1\}) = (\{1\}, \{1\})$ is not a restriction of $\overline{LR}(H) = (\{n\}, A)$, so $\overline{LR}$ is not monotonic. $\square$

To prove the positive results we identify first the crucial property of the initial game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ that holds in the presence of property **B**.

Lemma 2 *Assume property **B**. Then for all relaxations R of $\overline{LR}$ and all $i \in [1..n]$*

$$\forall \alpha \, \forall \mu_i \in \mathcal{B}_i \, \dot\cap \, R^\alpha \, \forall s_i \in T_i \, (s_i \in BR_{R^\alpha}(\mu_i) \to s_i \in BR_H(\mu_i)),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

In words, for all α and all beliefs in $\mu_i \in \mathcal{B}_i \dot\cap R^\alpha$ if s_i is a best response to μ_i in R^α, then it is in fact a best response to μ_i in H.

Proof. Suppose $s_i \in BR_{R^\alpha}(\mu_i)$, where $\mu_i \in \mathcal{B}_i \dot\cap R^\alpha$. The operator $\overline{LR}$ is contracting, so by the second condition in the definition of a relaxation for all $\beta \leq \alpha$ we have $R^\alpha \subseteq R^\beta$. Therefore by property **A** for all $\beta \leq \alpha$ we have $\mu_i \in \mathcal{B}_i \dot\cap R^\beta$. In particular $\mu_i \in \mathcal{B}_i$.

By property **B** a best response s_i^* to μ_i in H exists. But $s_i^* \in BR_H(\mu_i)$ implies that for all $\beta \leq \alpha$, $s_i^* \in BR_{R^\beta}(\mu_i)$, so for all $\beta \leq \alpha$, $s_i^* \in \overline{LR}(R^\beta) \subseteq R^{\beta+1}$ (the inclusion holds since R is a relaxation of $\overline{LR}$). Moreover $s_i^* \in R^0 = H$ and if $s_i^* \in R^\beta$ for all successor ordinals $\beta \leq \alpha$, then also $s_i^* \in R^\beta$ for all limit

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ordinals $\beta \leq \alpha$. So s_i^* is a strategy of player i in all restrictions R^β for $\beta \leq \alpha$, in particular R^α. Hence

$$p_i(s_i^*, \mu_i) = p_i(s_i, \mu_i).$$

So s_i is also a best response to μ_i in H. $\qquad\qquad\qquad\qquad\qquad\square$

We prove now the following positive results under the assumption of property **B**. Part (i) shows that arbitrary iterations of the $\overline{GR}$ and $\overline{LR}$ operators coincide. Part (ii) establishes order independence of the operator $\overline{LR}$. This is not an immediate consequence of part (i) since in general the relaxations of $\overline{GR}$ and $\overline{LR}$ differ. Finally, in part (iii) we return to the LR operator that we considered and rejected in the previous section.

Theorem 3 *Assume properties* **A** *and* **B**.

(i) *For all α we have $\overline{GR}^\alpha = \overline{LR}^\alpha$. In particular $\overline{GR}^{\alpha_{\overline{GR}}} = \overline{LR}^{\alpha_{\overline{LR}}}$.*

(ii) *The $\overline{LR}$ operator is order independent.*

(iii) *For all α we have $LR^\alpha = \overline{LR}^\alpha$. In particular $LR^{\alpha_{LR}}$ exists and $LR^{\alpha_{LR}} = \overline{LR}^{\alpha_{\overline{LR}}}$.*

Proof.
(i) For all G we have $\overline{GR}(G) \subseteq \overline{LR}(G)$, so in view of Lemma 1 we only need to prove for all α the inclusion $\overline{LR}^\alpha \subseteq \overline{GR}^\alpha$. We proceed by transfinite induction. Suppose that $\overline{LR}^\alpha \subseteq \overline{GR}^\alpha$.

Let s_i be a strategy of player i in $\overline{LR}^{\alpha+1}$. By definition for some $\mu_i \in \mathcal{B}_i \dot\cap \overline{LR}^\alpha$ we have $s_i \in BR_{\overline{LR}^\alpha}(\mu_i)$, so by Lemma 2 $s_i \in BR_H(\mu_i)$. Further, $\overline{LR}^{\alpha+1} \subseteq \overline{LR}^\alpha$, so by the induction hypothesis s_i is a strategy of player i in $\overline{GR}^\alpha$. So s_i is a strategy of player i in $\overline{GR}^{\alpha+1}$.

The induction step for a limit ordinal is immediate.

(ii) Consider a relaxation R of $\overline{LR}$ that has an outcome. R^{α_R} is a fixpoint of R, so R^{α_R} is also a fixpoint of $\overline{LR}$. Let s_i be a strategy of player i in R^{α_R}. By the definition of $\overline{LR}$ for some $\mu_i \in \mathcal{B}_i \dot\cap R^{\alpha_R}$ we have $s_i \in BR_{R^{\alpha_R}}(\mu_i)$, so by Lemma 2 $s_i \in BR_H(\mu_i)$. Since s_i was arbitrary, this proves that R^{α_R} is a fixpoint of $\overline{GR}$. Hence by Tarski's Fixpoint Theorem $R^{\alpha_R} \subseteq \overline{GR}^{\alpha_{\overline{GR}}}$.

Moreover, by Lemma 1 $\overline{GR}^{\alpha_{\overline{GR}}} \subseteq R^{\alpha_R}$. Consequently $R^{\alpha_R} = \overline{GR}^{\alpha_{\overline{GR}}}$.

(iii) We proceed by transfinite induction. Suppose the claim holds for all $\beta \leq \alpha$. To prove it for $\alpha + 1$ we first we prove that $LR^{\alpha+1} \subseteq LR^\alpha$. Let s_i

be a strategy of player i in $LR^{\alpha+1}$. Then for some $\mu_i \in \mathcal{B}_i \mathbin{\dot\cap} LR^\alpha$ we have $s_i \in BR_{LR^\alpha}(\mu_i)$. By the induction hypothesis $LR^\alpha = \overline{LR}^\alpha$, so by Lemma 2 $s_i \in BR_H(\mu_i)$.

Further, since $\overline{LR}$ is contracting, by property **A** for all $\beta \le \alpha$ we have $\mu_i \in \mathcal{B}_i \mathbin{\dot\cap} \overline{LR}^\beta$ and hence by the induction hypothesis for all $\beta \le \alpha$ we have $\mu_i \in \mathcal{B}_i \mathbin{\dot\cap} LR^\beta$. So s_i is a strategy of player i in all restrictions LR^β for $\beta \le \alpha$, in particular LR^α. Since s_i was arbitrary, this proves that $LR^{\alpha+1} \subseteq LR^\alpha$.

Hence

$$LR^{\alpha+1} = LR^{\alpha+1} \cap LR^\alpha = \overline{LR}(LR^\alpha) = \overline{LR}^{\alpha+1},$$

where the last equality holds by the induction hypothesis.

The induction step for a limit ordinal is immediate. $\qquad\square$

We mentioned already that property **B** is satisfied by the compact and continuous games and pure belief structures. So part (i) is closely related to the result of Ambroszkiewicz [1994] who showed that for such games the finite iterations of his versions of $\overline{GR}$ and $\overline{LR}$ operators coincide for the limited case of two-player games and pure belief structures. In his definition both operators are defined by considering the reduction for each player in succession and not in parallel.

In summary, by virtue of the results established in the last three sections, in the presence of property **B**, rationalizability can be equivalently defined using any of the introduced four operators $GR, \overline{GR}, LR$ and $\overline{LR}$. However, when property **B** does not hold, only the first two operators are of interest.

7   Strict dominance: the GS and $\overline{GS}$ operators

Recall that one of the consequences of the assumption that all players are rational is that none of them could possibly use any strictly dominated strategy. In this and the next section we apply our general approach to operators on games to establish analogous results for the notion strict dominance.[4] In what follows we analyze four operators that can be naturally used to define iterated elimination of strictly dominated strategies.

Given a restriction $G := (S_1, \ldots, S_n)$ of $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and two strategies s_i, s_i' from T_i we write $s_i' \succ_G s_i$ as an abbreviation for the statement $\forall s_{-i} \in S_{-i}\ p_i(s_i', s_{-i}) > p_i(s_i, s_{-i})$ and say then that s_i' **strictly dominates** s_i **on** G or, equivalently, that s_i **is strictly dominated on** G **by** s_i'.

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[4]As is well-known there is an intimate connection between the notions of rationalizability and strict dominance. This, however, has no bearing on our results. We shall discuss this matter in Section 10.

First, we introduce the following operator GS (standing for 'global strict dominance') on the set of restrictions of H:

$$GS(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg \exists s'_i \in T_i \; s'_i \succ_G s_i\}.$$

So $GS(G)$ is obtained by removing from H all strategies that are strictly dominated on G by some strategy in H and *not* in G. The reasoning embodied in the definition of this operator is analogous to the one used for GR: we remove a strategy from H if a 'better' (here: strictly dominating) strategy exists in the initial game H, *even* if in the iteration process leading from H to G this better strategy might have been removed.

The GS operator was introduced in Milgrom and Roberts [1990], where only its iterations up to ω were considered. As noted there this operator is clearly monotonic. So Note 1 applies and we can confine our considerations to the $\overline{GS}$ operator (defined by $\overline{GS}(G) := GS(G) \cap G$) that is also monotonic.[5] On the account of the results of Section 3 this operator enjoys the same properties as the $\overline{GR}$ operator. We summarize them in the theorem below.

Theorem 4 ($\overline{GS}$)

(i) *The largest fixpoint of $\overline{GS}$ exists and is its outcome.*

(ii) *$\overline{GS}$ is order independent.*

(iii) *For all relaxations R of $\overline{GS}$ and all α we have $\overline{GS}^\alpha \subseteq R^\alpha$.*   □

The $\overline{GS}$ operator was introduced and analyzed in Chen, Long and Luo [2005], where the need for transfinite iterations was explained and where properties (i) and (ii) were proved.

8   Strict dominance: the LS and $\overline{LS}$ operators

The customary definition of the iterated elimination of strictly dominated strategies (IESDS) involves elimination of strategies that are strictly dominated by a strategy from the *currently considered* game and *not* the initial

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[5]The reader may now notice that in Example 2 we used the $\overline{GS}$ operator instead of GS, which is more complicated to justify informally.

game H. A natural operator that formalizes this idea is the following one (LS stands for 'local strict dominance'):

$$LS(G) := (S'_1, \ldots, S'_n),$$

where $G := (S_1, \ldots, S_n)$ and for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg \exists s'_i \in S_i \; s'_i \succ_G s_i\}.$$

So here, in contrast to GS, a strategy is removed if it is strictly dominated on G by some strategy *in G* itself. However, the LS operator is not acceptable for the same reason as the previously considered LR operator.

Example 9 Reconsider the two-player game H from Example 5 in which the set of strategies for each player is the set of natural numbers. The payoff to each player is the number he selected. In this game each strategy is strictly dominated.

So $LS(H) = (\emptyset, \emptyset)$. But $LS((\emptyset, \emptyset)) = H$, since the condition $\neg \exists s'_i \in S_i s'_i \succ_G s_i$ is vacuously satisfied when $S_i = \emptyset$. So the iterations of LS oscillate between H and $(\emptyset, \emptyset)$. Hence the outcome of LS does not exist.

A fortiori LS is not monotonic and not contracting. The most problematic is of course that this operator yields a non-empty restriction (in fact, the initial game) when applied to an empty restriction. □

So, as in the case of the LR operator, we confine our attention to the contracting version $\overline{LS}$ (defined by $\overline{LS}(G) := LS(G) \cap G$). In fact, in the literature the LS operator was not considered but rather $\overline{LS}$. However, $\overline{LS}$ fails to be monotonic, even for finite games. Indeed, the game from Example 8 provides the evidence.

The $\overline{LS}$ operator was shown to be order independent for finite games in Gilboa, Kalai and Zemel [1990]. It was studied for infinite games in Dufwenberg and Stegeman [2002] where it was noted that it fails to be order independent for arbitrary games. This is immediate to see using the game from Examples 5 and 9. Indeed, the outcome of $\overline{LS}$ is then $(\emptyset, \emptyset)$, whereas for any $i \geq 0$ the restriction $(\{i\}, \{i\})$ is the outcome of a relaxation R of $\overline{LS}$ such that $R(H) = (\{i\}, \{i\})$.

In the remainder of this section we prove a limited form of order independence of $\overline{LS}$. Then, in the next section, we compare our results with those of Dufwenberg and Stegeman [2002]. First, following Dufwenberg and Stegeman [2002], we consider the following property of the initial game H:

P For all relaxations R of $\overline{LS}$ and all $k \geq 0$, every strictly dominated strategy in R^k has an undominated ('best') dominator.

This property limited to the initial game (so for $k = 0$) was considered in Milgrom and Roberts [1996]. We generalize this property to all ordinals and formalize it as follows:

C(α) For all relaxations R of $\overline{LS}$ and all $i \in [1..n]$

$$\forall s_i \in T_i \ (\exists s_i' \in T_i \ s_i' \succ_{R^\alpha} s_i \rightarrow$$
$$\exists s_i^* \in T_i \ (s_i^* \succ_{R^\alpha} s_i \wedge \neg \exists s_i' \in T_i \ s_i' \succ_{R^\alpha} s_i^*)).$$

Note that all the quantifiers range over T_i and not S_i. So we refer to strict dominance by a strategy in the initial game H and *not* in the currently considered reduction R^α. In what follows we shall rather use the following simpler property:

D(α) For all relaxations R of $\overline{LS}$ and all $i \in [1..n]$

$$\forall s_i \in T_i((\exists s_i' \in T_i \ s_i' \succ_{R^\alpha} s_i) \rightarrow \exists s_i^* \in S_i \ s_i^* \succ_{R^\alpha} s_i),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

It states that each s_i strictly dominated in R^α is in fact strictly dominated in R^α by some strategy in R^α.

First we establish a lemma that clarifies the relation between these two properties.

Lemma 3 *For each α, **C(α)** implies **D(α)**.*

Proof. Consider a relaxation R of $\overline{LS}$, some $i \in [1..n]$. Let $R^\alpha := (S_1, \ldots, S_n)$. Suppose $s_i \in T_i$ and $s_i' \in T_i$ are such that $s_i' \succ_{R^\alpha} s_i$. By property **C($\alpha$)**

$$\exists s_i^* \in T_i \ (s_i^* \succ_{R^\alpha} s_i \wedge \neg \exists s_i' \in T_i \ s_i' \succ_{R^\alpha} s_i^*).$$

$\overline{LS}$ is contracting so for $\beta \leq \alpha$ we have $R^\alpha \subseteq R^\beta$ and consequently $\forall \beta \leq \alpha \ \neg \exists s_i' \in T_i \ s_i' \succ_{R^\beta} s_i^*$. Hence s_i^* is a strategy of player i in all restrictions R^β for $\beta \leq \alpha$. In particular $s_i^* \in S_i$. $\qquad \square$

Consequently all the results that follow also hold when property **C(α)** is used instead of **D(α)**.

We now establish a number of consequences of property **D(α)** for various values of α. Part (i) is a counterpart of Theorem 3(i). Part (ii) is a limited order independence result for the $\overline{LS}$ operator. Finally, in part (iii) we compare the operators LS and $\overline{LS}$ and establish an analogue of Theorem 3(iii).

Theorem 5

(i) *Assume property $\forall \beta < \alpha\, \mathbf{D}(\beta)$ for an ordinal α. Then $\overline{GS}^{\alpha} = \overline{LS}^{\alpha}$. In particular, if $\forall \alpha\, \mathbf{D}(\alpha)$, then $\overline{GS}^{\alpha_{GS}} = \overline{LS}^{\alpha_{LS}}$.*

(ii) *Assume property $\forall \alpha\, \mathbf{D}(\alpha)$. Then the $\overline{LS}$ operator is order independent.*

(iii) *Assume property $\forall \beta < \alpha + 1\, \mathbf{D}(\beta)$. Then $LS^{\alpha} = \overline{LS}^{\alpha}$. In particular, if $\forall \alpha\, \mathbf{D}(\alpha)$, then $LS^{\alpha_{LS}}$ exists and $LS^{\alpha_{LS}} = \overline{LS}^{\alpha_{LS}}$.*

Proof.
(i) In view of Lemma 1 we only need to prove the inclusion $\overline{LS}^{\alpha} \subseteq \overline{GS}^{\alpha}$ and only the induction step for a successor ordinal requires a justification. So suppose that $\forall \beta < \alpha + 1\, \mathbf{D}(\beta)$. Let $\overline{LS}^{\alpha} := (S_1, \ldots, S_n)$.

Let s_i be a strategy of player i in $\overline{LS}^{\alpha+1}$. By definition $s_i \in S_i$ and $\neg \exists s_i' \in S_i \ s_i' \succ_{\overline{LS}^{\alpha}} s_i$, so by the assumed property $\neg \exists s_i' \in T_i \ s_i' \succ_{\overline{LS}^{\alpha}} s_i$. Hence $\neg \exists s_i' \in T_i s_i' \succ_{\overline{GS}^{\alpha}} s_i$, since by the induction hypothesis $\overline{LS}^{\alpha} \subseteq \overline{GS}^{\alpha}$. Also, because of the same inclusion, s_i is a strategy of player i in $\overline{GS}^{\alpha}$. So s_i is a strategy of player i in $\overline{GS}^{\alpha+1}$.

(ii) The proof is similar to that of Theorem $3(ii)$. Consider a relaxation R of $\overline{LS}$ that has an outcome. R^{α_R} is a fixpoint of R, so R^{α_R} is also a fixpoint of $\overline{LS}$.

Suppose now $R^{\alpha_R} := (S_1, \ldots, S_n)$ and take some $s_i \in S_i$. By the definition of $\overline{LS}$ we have $\neg \exists s_i^* \in S_i \ s_i^* \succ_{R^{\alpha_R}} s_i$, so by property $\mathbf{D}(\alpha_R)$ we get $\neg \exists s_i' \in T_i \ s_i' \succ_{R^{\alpha_R}} s_i$. Since s_i was arbitrary, this proves that R^{α_R} is a fixpoint of $\overline{GS}$. Hence by Tarski's Fixpoint Theorem $R^{\alpha_R} \subseteq \overline{GS}^{\alpha_{GS}}$.

Moreover, for all G we have both $\overline{GS}(G) \subseteq \overline{LS}(G)$ and $\overline{LS}(G) \subseteq R(G)$, so by Lemma 1 $\overline{GS}^{\alpha_{GS}} \subseteq R^{\alpha_R}$ and consequently $R^{\alpha_R} = \overline{GS}^{\alpha_{GS}}$.

(iii) We prove the claim by transfinite induction. Assume it holds for all $\beta \leq \alpha$. We prove it for $\alpha + 1$. So suppose that $\forall \beta < \alpha + 2\, \mathbf{D}(\beta)$. Let $LS^{\alpha} := (S_1, \ldots, S_n)$. Let s_i be a strategy of player i in $LS^{\alpha+1}$. By definition

$$\neg \exists s_i' \in S_i \ s_i' \succ_{LS^{\alpha}} s_i.$$

By the induction hypothesis $LS^{\alpha} = \overline{LS}^{\alpha}$, so by the assumed property $\neg \exists s_i' \in T_i \ s_i' \succ_{\overline{LS}^{\alpha}} s_i$. So s_i is a strategy of player i in $GS(\overline{LS}^{\alpha})$. But by part (i) $\overline{LS}^{\alpha} = \overline{GS}^{\alpha}$ and $\overline{LS}^{\alpha+1} = \overline{GS}^{\alpha+1}$. Also, since GS is monotonic, by Note 1 $\overline{GS}^{\alpha} = GS^{\alpha}$ and $\overline{GS}^{\alpha+1} = GS^{\alpha+1}$. So

$$GS(\overline{LS}^{\alpha}) = GS(\overline{GS}^{\alpha}) = GS^{\alpha+1} = \overline{LS}^{\alpha+1}.$$

Since s_i was arbitrary, this shows that $LS^{\alpha+1} \subseteq \overline{LS}^{\alpha+1}$. But also

$$\overline{LS}^{\alpha+1} = \overline{LS}(LS^\alpha) = LS^{\alpha+1} \cap LS^\alpha \subseteq LS^{\alpha+1},$$

where the first equality holds by the induction hypothesis. So $\overline{LS}^{\alpha+1} = LS^{\alpha+1}$. The induction step for a limit ordinal is immediate. $\square$

Property $\forall\alpha \mathbf{C}(\alpha)$ obviously holds when the initial game is finite and implies by Lemma 3 property $\forall\alpha \mathbf{D}(\alpha)$, so part (ii) generalizes the already mentioned result of Gilboa, Kalai and Zemel [1990]. Further, note that in the proof of part (i) we actually use a weaker property than $\forall\beta < \alpha \mathbf{D}(\beta)$. Indeed, we use it only with R equal to $\overline{LS}$, so it suffices to assume that for all $\beta < \alpha$

$$\forall s_i \in T_i((\exists s_i' \in T_i\; s_i' \succ_{\overline{LS}^\beta} s_i) \rightarrow \exists s_i^* \in S_i\; s_i^* \succ_{\overline{LS}^\beta} s_i),$$

where $\overline{LS}^\beta := (S_1, \ldots, S_n)$.

In summary, the iterations of two operators, GS and $\overline{GS}$, always coincide and each of them is order independent. Further, when $\forall\alpha \mathbf{D}(\alpha)$ holds, all iterations of all four operators, $GS, \overline{GS}, LS$ and $\overline{LS}$, coincide and any of them can be used to define the outcome of IESDS. The property $\forall\alpha \mathbf{D}(\alpha)$ holds when the initial game is finite, so to conclude this section let us summarize the consequences of the above result for this case.

Corollary 1 *Suppose the initial game H is finite. Then*

(i) for all $k \geq 0$ the iterations of all four operators coincide, i.e.,

$$GS^k = \overline{GS}^k = \overline{LS}^k = LS^k,$$

(ii) the outcomes of these four operators exist and coincide,

(iii) the operators GS, $\overline{GS}$ and $\overline{LS}$ are order independent. $\square$

9 Comparison with the results of Dufwenberg and Stegeman

We now compare the results of the previous section with those of Dufwenberg and Stegeman [2002]. First, we introduce the following notion. We say that

G is an ω-**outcome** of an operator T on the set of restrictions of H if G is an outcome of T and $\alpha_T \leq \omega$.[6]

That is, G is an ω-outcome of T if $G = T^\omega$ and $T^{\omega+1} = T^\omega$. For a contracting operator, in contrast to the outcome, an ω-outcome does not need to exist. The study of Dufwenberg and Stegeman [2002] focuses on the set

$$\omega(\overline{LS}) := \{G \mid G \text{ is an } \omega\text{-outcome of a relaxation of } \overline{LS}\}.$$

If this set has at most one element, then they view $\overline{LS}$ as order independent. In what follows we refer then to **DS-order independence**. Recall that according to our definition an operator T is order independent if the set

$$\{G \mid G \text{ is an outcome of a relaxation of } T\}$$

has at most one element. We can then state the main results of Dufwenberg and Stegeman [2002] as follows. To clarify part (i) recall that by definition a restriction $(S_1, \ldots, S_n)$ is a game if each S_i is non-empty.

Theorem 6

(i) *If H is compact and own-uppersemicontinuous and the set $\omega(\overline{LS})$ has an element which is a game, then this is its only element.*

(ii) *If H is compact and continuous, then the set $\omega(\overline{LS})$ has precisely one element and this element is a compact and continuous game.* □

Dufwenberg and Stegeman [2002] considered property **P** from the previous section but formalized it as $\forall k < \omega \mathbf{E}(k)$, where for an ordinal α we have:

$\mathbf{E}(\alpha)$ For all relaxations R of $\overline{LS}$ and all $i \in [1..n]$

$$\forall s_i \in S_i \ (\exists s_i' \in S_i \ s_i' \succ_{R^\alpha} s_i \rightarrow$$
$$\exists s_i^* \in S_i \ (s_i^* \succ_{R^\alpha} s_i \wedge \neg \exists s_i' \in S_i \ s_i' \succ_{R^\alpha} s_i^*)),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

They showed that property $\forall k < \omega \mathbf{E}(k)$ is satisfied by the compact and own-uppersemicontinuous games but is not a sufficient condition for DS-order independence of $\overline{LS}$. Note that the difference between the properties $\mathbf{C}(\alpha)$ and $\mathbf{E}(\alpha)$ is that in the former all the quantifiers range over T_i and not S_i. So in

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[6]In Dufwenberg and Stegeman [2002] an ω-outcome is called a **maximal** ($\rightarrow^*$)-**reduction**.

$\mathbf{C}(\alpha)$ we refer to strict dominance by a strategy in the initial game H, while in $\mathbf{E}(\alpha)$ to strict dominance by a strategy in the currently considered reduction R^α.

The following result then relates property $\mathbf{D}(\alpha)$ (and hence by Lemma 3 property $\mathbf{C}(\alpha)$) to DS-order independence of $\overline{LS}$.

Theorem 7 *Assume property $\mathbf{D}(\omega)$. Then $\omega(\overline{LS}) \subseteq \{\overline{LS}^\omega\}$, so $\overline{LS}$ is DS-order independent.*

Proof. Let R be a relaxation of $\overline{LS}$ such that $\alpha_R \leq \omega$. For all α we have both $\overline{GS}(R^\alpha) \subseteq \overline{LS}(R^\alpha)$ and $\overline{LS}(R^\alpha) \subseteq R(R^\alpha)$, so by Lemma 1

$$\overline{GS}^\omega \subseteq R^\omega. \tag{1}$$

Let $R^\omega := (S_1, \ldots, S_n)$. Take a strategy $s_i \in S_i$. R^ω is a fixpoint of R, so it is a fixpoint of $\overline{LS}$. Hence $\neg \exists s_i' \in S_i \ s_i' \succ_{R^\omega} s_i$, so by property $\mathbf{D}(\omega)$ $\neg \exists s_i' \in T_i \ s_i' \succ_{R^\omega} s_i$. Since s_i was arbitrary, this shows that R^ω is a fixpoint of $\overline{GS}$. So by the $\overline{GS}$ Theorem 4(i) we have $R^\omega \subseteq \overline{GS}^{\alpha_{\overline{GS}}}$. Also, $\overline{GS}$ is contracting, so $\overline{GS}^{\alpha_{\overline{GS}}} \subseteq \overline{GS}^\omega$ and hence

$$R^\omega \subseteq \overline{GS}^\omega.$$

This inclusion combined with (1) yields $R^\omega = \overline{GS}^\omega$. By Theorem 5$(i)$ $R^\omega = \overline{LS}^\omega$.

So we showed that any ω-outcome of a relaxation of $\overline{LS}$ equals $\overline{LS}^\omega$. This concludes the proof. □

Dufwenberg and Stegeman [2002] also showed that for compact and own-uppersemicontinuous games property $\forall k \leq \omega \mathbf{D}(k)$ holds. So the above theorem is a minor improvement of the result of Dufwenberg and Stegeman [2002] listed earlier as Theorem 6(i). Finally, note that when $\forall k \leq \omega \mathbf{D}(k)$ holds, then by Theorem 5(i) and (iii) the first ω iterations of all four operators, $GS, \overline{GS}, LS$ and $\overline{LS}$, coincide. So in particular this is the case for the compact and own-uppersemicontinuous games.

10 Strict dominance by a mixed strategy

It is well-known that the notions of strict dominance and best responses, and consequently the notions of iterated elimination of strictly dominated strategies and of rationalizability, are closely related. In this context one usually

considers strict dominance by a mixed strategy. In section we review this relationship and clarify to what extent the results of the previous three sections can be extended to mixed strategies.

In Moulin [1984] the class of so-called **nice games** is introduced for which the notions of a best response to a point belief (i.e., a joint pure strategy of the opponents) and of not being weakly dominated by a pure strategy coincide. These are games $(T_1, \ldots, T_n, p_1, \ldots, p_n)$ in which each strategy set T_i is a compact and convex subset of $\mathcal{R}$ and each payoff function p_i is continuous and **strictly quasiconcave** w.r.t. T_i, where the latter means that for all $\alpha \in (0,1)$ and $s_i, s_i' \in T_i$ with $s_i \neq s_i'$ and all $s_{-i} \in T_{-i}$

$$p_i(s_i, s_{-i}) \geq p_i(s_i', s_{-i}) \text{ implies } p_i(\alpha s_i + (1-\alpha)s_i', s_{-i}) > p_i(s_i', s_{-i}).$$

In Zimper [2006] it is clarified that for nice games the notions of weak dominance and strict dominance coincide. This yields for nice games equivalence between the notions of a best response and of not being strictly dominated, both w.r.t. pure strategies. This result is then generalized by assuming instead of strict quasiconcativity that each payoff function p_i is **quasiconcave** w.r.t. T_i, which means that for all $\alpha \in (0,1)$ and $s_i, s_i' \in T_i$ with $s_i \neq s_i'$ and all $s_{-i} \in T_{-i}$

$$p_i(s_i, s_{-i}) > p_i(s_i', s_{-i}) \text{ implies } p_i(\alpha s_i + (1-\alpha)s_i', s_{-i}) > p_i(s_i', s_{-i})$$

and

$$p_i(s_i, s_{-i}) = p_i(s_i', s_{-i}) \text{ implies } p_i(\alpha s_i + (1-\alpha)s_i', s_{-i}) \geq p_i(s_i', s_{-i}).$$

It is also shown that the equivalence does not need to hold anymore if the strategy sets T_i are subsets of $\mathcal{R}^2$ instead of $\mathcal{R}$.

In the case of finite games the notion of strict dominance extends in the obvious to the case of mixed strategies. By the result of Pearce [1984] in two-player finite games the notions of a best response to a mixed strategy of the opponent and of not being strictly dominated by a mixed strategy coincide. In Osborne and Rubinstein [1994] this result is presented as a result for n-players finite games where the best response is defined w.r.t. correlated mixed strategies. More recently, in Zimper [2005], both results were generalized to compact games with bounded and continuous payoff functions.

In an arbitrary game $(T_1, \ldots, T_n, p_1, \ldots, p_n)$ the set of mixed strategies ΔT_i of player i is defined as the set of *all* probability measures on a given σ-algebra of subsets of T_i. In the case of compact games with continuous payoff functions it is customary (as in Zimper [2005]) to take the σ-algebra of Borel sets. The

payoff functions are then extended from pure to mixed strategies in a standard way using integration. To ensure that the payoffs remain finite the original payoff functions are assumed to be bounded.

If the payoff functions are unbounded, an alternative (used in Example 5) is to define mixed strategies as the probability measures with a finite support. More general approaches for two-player games are studied in Tijs [1975].

In what follows we just assume that given the initial game $(T_1, \ldots, T_n, p_1, \ldots, p_n)$ for each $i \in [1..n]$ a set $\Delta T_i \supseteq T_i$ of mixed strategies of player i is given and that each payoff function p_i is extended to $p_i : \Delta T_1 \times \ldots \times \Delta T_n \to \mathcal{R}$. If the initial game is finite, we take the usual set of mixed strategies and the customary extension of each payoff function.

Then the results of Section 8 can be directly adapted to the case of strict dominance by a mixed strategy as follows. First, we introduce the counterpart of the LS operator defined by:

$$MLS(G) := (S'_1, \ldots, S'_n),$$

where $G := (S_1, \ldots, S_n)$ and for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg \exists m_i \in \Delta S_i \ m_i \succ_G s_i\},$$

where we use the extension of the $\succ_G$ relation to the pairs of mixed and pure strategies.

In the literature, in the case of finite games, the iterated elimination of strategies that are strictly dominated by a mixed strategy is defined as the iteration of $\overline{MLS}$, the contracting version of the above operator. The obvious modification of the GS operator to the mixed strategies is defined by:

$$MGS(G) := (S'_1, \ldots, S'_n),$$

where for all $i \in [1..n]$

$$S'_i := \{s_i \in T_i \mid \neg \exists m_i \in \Delta T_i \ m_i \succ_G s_i\}.$$

Just as GS, the MGS operator is clearly monotonic. Its contracting version, $\overline{MGS}$, is studied in Brandenburger, Friedenberg and Keisler [2006] (it is their map Φ), where its relation to the concept of best response sets is clarified.

Next, we modify the property $\mathbf{D}(\alpha)$ to the case of mixed strategies:

$\mathbf{MD}(\alpha)$ For all relaxations R of $\overline{MLS}$ and all $i \in [1..n]$

$$\forall s_i \in T_i((\exists m_i \in \Delta T_i \ m_i \succ_{R^\alpha} s_i) \to \exists m_i^* \in \Delta S_i \ m_i^* \succ_{R^\alpha} s_i),$$

where $R^\alpha := (S_1, \ldots, S_n)$.

Then we have the following direct counterpart of Theorem 5.

Theorem 8

(i) *Assume property $\forall \beta < \alpha \mathbf{MD}(\beta)$ for an ordinal α. Then $\overline{MGS}^{\alpha} = \overline{MLS}^{\alpha}$. In particular, if $\forall \alpha \mathbf{MD}(\alpha)$, then $\overline{MGS}^{\alpha_{\overline{MGS}}} = \overline{MLS}^{\alpha_{\overline{MLS}}}$.*

(ii) *Assume property $\forall \alpha \mathbf{MD}(\alpha)$. Then the $\overline{MLS}$ operator is order independent.*

(iii) *Assume property $\forall \beta < \alpha + 1 \ \mathbf{MD}(\beta)$. Then $MLS^{\alpha} = \overline{MLS}^{\alpha}$. In particular, if $\forall \alpha \mathbf{MD}(\alpha)$, then $MLS^{\alpha_{MLS}}$ exists and $MLS^{\alpha_{MLS}} = \overline{MLS}^{\alpha_{\overline{MLS}}}$.*

Proof. Analogous to the proof of Theorem 5 and omitted. $\square$

Consequently, when $\forall \alpha \mathbf{MD}(\alpha)$ all iterations of all four operators, MGS, $\overline{MGS}$, MLS and $\overline{MLS}$, coincide. To establish this result, as in the case of strict dominance by a pure strategy, it is sufficient to use property $\mathbf{MD}(\alpha)$ with R equal to $\overline{MLS}$.

The question remains for which games property $\mathbf{MD}(\alpha)$ holds. We found (the details are relegated to another paper) that in the case of finite games and the customary set of mixed strategies, property $\mathbf{MD}(k)$ holds for all $k \geq 0$. Since the closure ordinals of the above four operators are then finite, we conclude by the above theorem that all iterations of the above four operators then coincide.

11 Concluding remarks

In this paper we analyzed two widely used ways of reducing strategic games concerned with the concepts of rationalizability and iterated elimination of strictly dominated strategies. We showed that both concepts can be defined in a number of ways that differ for arbitrary infinite games. Also, we clarified for which games these differences disappear. Our analysis was based on a general study of operators on complete lattices and showed that concepts defined by means of monotonic operators are easier to assess and study.

In some circumstances a reduction notion defined using a non-monotonic operator still can be analyzed using our approach, by relating the operator to a monotonic one. This is for example how we established the order independence of the $\overline{LS}$ operator under the assumption $\forall \alpha \mathbf{D}(\alpha)$ —by relating it to the $\overline{GS}$ operator which is monotonic.

An important aspect of our analysis is that we allow transfinite iterations of the corresponding operators. Their use in an analysis of reasoning used by rational agents can be baffling. These matters were originally discussed in Lipman [1991], where a need for transfinite iterations in the definition of rationalizability was noted. The distinction between finitary and infinitistic forms of reasoning is well understood in mathematical logic. Both forms have been widely used and we see no reason for limiting the study of games to finitary methods.

The final matter that merits attention is the striking difference between the way rationalizability and iterated elimination of strictly dominated strategies (IESDS) have been traditionally defined and used in the literature. Both concepts are supposed to capture reasoning used by rational players. Yet, in the definition of the former concept, according to Bernheim [1984], the reference point for a deletion of a strategy is the *initial* game, while in the definition of the latter one the *currently considered* game. This difference is important, since the first approach yields a monotonic operator, while the second one not.

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